Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Prove inequality

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le 12(2R-3r),$$

where a,b,c and m_a,m_b,m_c be, respectively, sides and medians of triangle $\triangle ABC$, with circumradius R and inradius r.

Solution.

Let d_a be distance from the circumcenter to side a. Then by triangle inequality $m_a \le R + d_a$ and, since $d_a = \sqrt{R^2 - \frac{a^2}{4}}$ then we obtain:

$$m_a - R \le \sqrt{R^2 - \frac{a^2}{4}} \iff m_a^2 - 2m_aR + R^2 \le R^2 - \frac{a^2}{4} \iff$$

(**1**) $\quad 4m_a^2 - 8m_aR + a^2 \le 0.$

Let F and s be area and semiperimeter, respectively and let h_a,h_b,h_c be regular notation for altitudes.

Using inequality (**1**), we obtain

$$m_a^2 + a^2 \le 8Rm_a \iff 4m_a + \frac{a^2}{m_a} \le 8R.$$

Hence, $\sum_{cyclic}\left(4m_a + \frac{a^2}{m_a}\right) \le 24R \iff \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le 24R - 4(m_a+m_b+m_c).$

From the other hand $m_a+m_b+m_c \ge h_a+h_b+h_c = 2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) =$

$\frac{F}{s}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \frac{9F}{s} = 9r.$

Thus, $\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le 24R - 4(m_a+m_b+m_c) \le 24R - 36r = 12(2R-3r).$ ■